CONDITION IN TERMS OF THE INVARIANTS OF THE QUARTIC THAT ITS FOUR DISTINCT ROOT-POINTS BE CONCYCLIC.

By DR. T. E. McKINNEY, Wesleyan University, Middletown, Conn.

The necessary and sufficient condition that the root-points of the quartic

(1) $$\sum_{i=0}^{i=4} a_i z^{4-i}, \quad a_i \text{ complex},$$

be concyclic is that an anharmonic ratio of the roots of the quartic be real. The equation giving the six anharmonic ratios of these four roots is

(2) $$t^6 - 3t^5 + \left(6 - \frac{I^3}{\Delta}\right)t^4 - \left(7 - \frac{2I^3}{\Delta}\right)t^3 + \left(6 - \frac{I^3}{\Delta}\right)t^2 - 3t + 1 = 0,$$

where $a_0^2 I = a_2^2 - 3a_1 a_3 + 12a_0 a_4$;

$$a_0^3 I_1 = 27(a_1^2 a_4 + a_0 a_3^2) - 9a_1 a_2 a_3 - 72a_0 a_2 a_4 + 2a_2^3,$$

$$27a_0^6 \Delta = 64I^3 - I_1^2.$$

The discriminant D of equation (2) is

$$D = \Pi(t_i - t_j)^2, \quad i=1, 2, \ldots, 5;\ j=i+1, \ldots, 6,$$

$$= \frac{I_1^2}{\Delta^4}\left(4\frac{I^3}{\Delta} - 27\right)^3.$$

Every root of equation (2) is a rational function with real coefficients of every other, so that the roots are either all real or all complex. When the roots are all real I^3/Δ is real. When the roots are not only real but also distinct, $D>0$. When I^3/Δ is real, and $D>0$, equation (2) has an even number of pairs of conjugate roots. Hence two and, therefore, all roots are real. This result may be expressed as follows:

THEOREM. *The necessary and sufficient condition that the four distinct root-points of the quartic*

$$\sum_{i=0}^{i=4} a_i z^{4-i}, \quad a_i \textit{ complex},$$

be concyclic is that $\frac{4I^3}{\Delta} - 27 > 0$, *where* I *and* Δ *are the invariants of the quartic.*